Exercise 9

Evaluate the line integral, where C is the given curve.

$$\int_C x^2 y\, ds, \quad C:\ x = \cos t,\ y = \sin t,\ z = t, \quad 0 \le t \le \pi/2$$

Solution

With the given parameterization, the line integral becomes

$$\begin{aligned}
\int_C x^2 y\, ds &= \int_0^{\pi/2} [x(t)]^2 y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\
&= \int_0^{\pi/2} (\cos t)^2 (\sin t) \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2}\, dt \\
&= \int_0^{\pi/2} \cos^2 t \sin t \sqrt{\sin^2 t + \cos^2 t + 1}\, dt \\
&= \sqrt{2} \int_0^{\pi/2} \cos^2 t \sin t\, dt.
\end{aligned}$$

Make the following substitution.

$$\begin{aligned}
u &= \cos t \\
du &= -\sin t\, dt \quad \rightarrow \quad -du = \sin t\, dt
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_C x^2 y\, ds &= \sqrt{2} \int_{\cos(0)}^{\cos(\pi/2)} u^2 (-du) \\
&= \sqrt{2} \int_1^0 u^2 (-du) \\
&= \sqrt{2} \int_0^1 u^2\, du \\
&= \sqrt{2} \left(\frac{u^3}{3}\right)\Bigg|_0^1 \\
&= \sqrt{2} \left(\frac{1}{3}\right) \\
&= \frac{\sqrt{2}}{3}.
\end{aligned}$$